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## Omae's knot and $12_{a990}$ are ribbon

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ABSTRACT. The purpose of this note is twofold: First, we prove that Omae's knot is ribbon, which was known to be homotopically slice. Second, we give a sufficient condition for a given knot to be ribbon. As a corollary, we show that the knot  $12_{a990}$  is ribbon, which was known to be slice.

### 1. OMAE'S KNOT IS RIBBON

A knot  $K$  in the 3-sphere  $S^3 = \partial D^4$  is *slice* if there exists a smoothly embedded disk  $D^2 \subset D^4$  such that  $\partial D^2 = K$ . A knot  $K$  is *ribbon* if there exists a smoothly immersed disk  $D^2 \subset S^3$  with only ribbon singularities such that  $\partial D^2 = K$ . It is easy to see that every ribbon knot is slice. The slice-ribbon conjecture due to Fox [5] states that every slice knot is ribbon, which has been a long-standing unsolved problem in knot theory.

In the positive direction, the slice-ribbon conjecture was conformed for two-bridge knots [19, Lisca], certain pretzel knots [11, Greene-Jabuka], certain Montesinos knots [17, Lecuona] and simple slice knots [23, Shibuya]. On the other hand, potential counterexamples to the slice-ribbon conjecture are demonstrated through the study of the 4-dimensional smooth Poincaré conjecture [2, 6, 7, 9].

Omae [22] studied the knot depicted in the left of Figure 1. The first author and Jong [1] observed that Omae's knot bounds a smoothly embedded disk in a homotopy 4-ball  $W$  which is represented by the handle diagram as in the right of Figure 1 (see also Section 4). In this note, we prove the following.

**Theorem 1.1.** *The 4-manifold  $W$  is diffeomorphic to the standard 4-ball.*

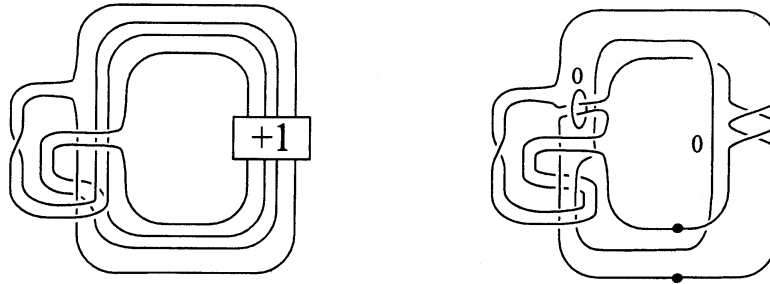


FIGURE 1. Omae's knot and a homotopy 4-ball  $W$ .

*Proof.* Handle calculus in Figure 2 implies that  $W$  is diffeomorphic to the standard 4-ball.  $\square$

**Corollary 1.2.** *Omae's knot is slice. Furthermore, it is ribbon.*

*Proof.* Theorem 1.1 implies that Omae's knot is slice. Recall that Omae's knot is isotopic to the boundary of cocore disk of the 2-handle (colored Grey) of the top left handle diagram in Figure 2. By chasing Omae's knot in handle diagrams in Figure 2, we obtain a ribbon presentation of Omae's knot as in Figure 3.  $\square$

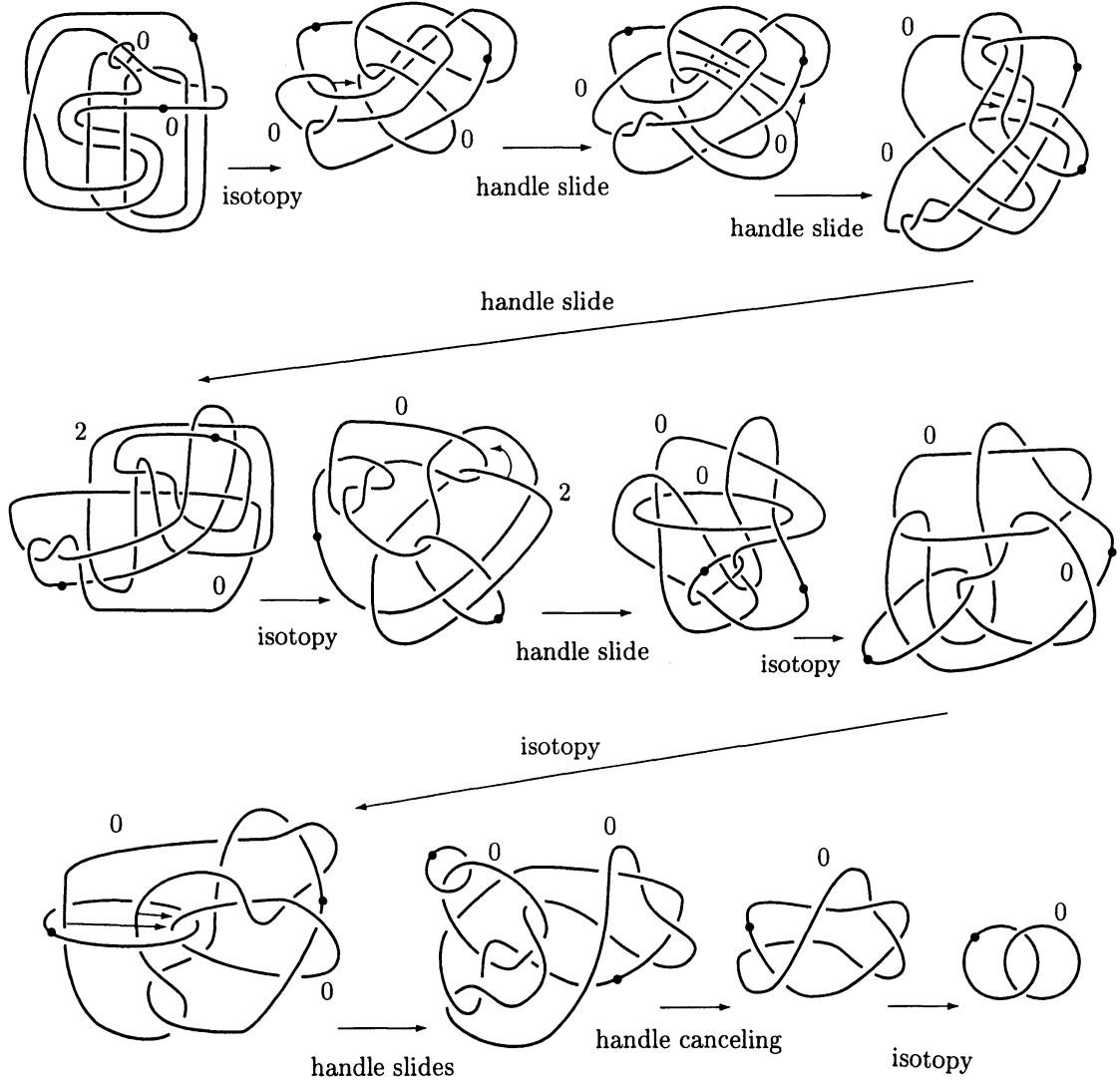
*Remark 1.3.* Another potential counterexample to the slice-ribbon conjecture is the  $(2, 1)$ -cable of the figure eight knot. Livingston and Melvin [18] and Kawauchi [14] proved that it is algebraically slice. Furthermore Kawauchi [15] showed that it is rationally slice. On the other hand, by the theorem of Casson-Gordon [4], Miyazaki [21] proved that it is not ribbon. Until now, it is not known whether the  $(2, 1)$ -cable of the figure eight knot is slice or not. See also Gomp-Miyazaki [8].

## 2. THE KNOT $12_{a990}$ IS RIBBON

The simplest slice knot which might not be ribbon is  $12_{a990}$ . Indeed, Herald, Kirk and Livingston [12] showed that the connected sum of  $12_{a990}$  and right- and left-handed trefoils is ribbon, implying that  $12_{a990}$  is slice. However it was unknown whether  $12_{a990}$  is ribbon<sup>1</sup>.

A  $t_n$ -move is a tangle replacement as in Figure 4. In this section, we show the following.

<sup>1</sup>C. Livingston (e-mail communication) informed us that they knew that  $12_{a990}$  is ribbon, however they did not write that  $12_{a990}$  is ribbon in [12].

FIGURE 2. Handle diagrams which represent  $W$ .

**Theorem 2.1.** *Let  $K$  be a knot. If we obtain the 3-component unlink from  $K$  by applying a  $t_{2n+1}$ - and  $t_{-(2n+1)}$ -move, then  $K$  is ribbon.*

We denote by  $T(p, q)$  the torus knot of type  $(p, q)$ . First, we show the following.

**Lemma 2.2.** *Let  $K$  be a knot. If we obtain the 3-component unlink from  $K$  by applying a  $t_{2n+1}$ - and  $t_{-(2n+1)}$ -move, then  $K \# T(2, 2n+1) \# T(2, -(2n+1))$  is ribbon, where  $\#$  denotes the connected sum.*

*Proof.* We may assume that a  $t_{2n+1}$ -move and a  $t_{-(2n+1)}$ -move are done simultaneously. In other words, there exist two trivial tangles  $(B_+, T_+)$  and

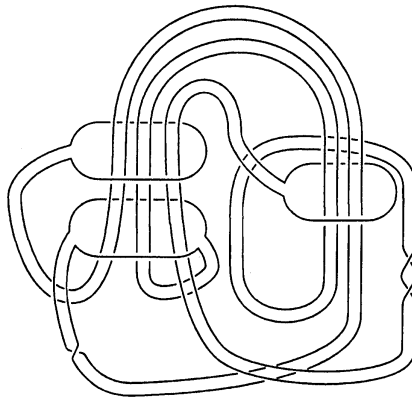
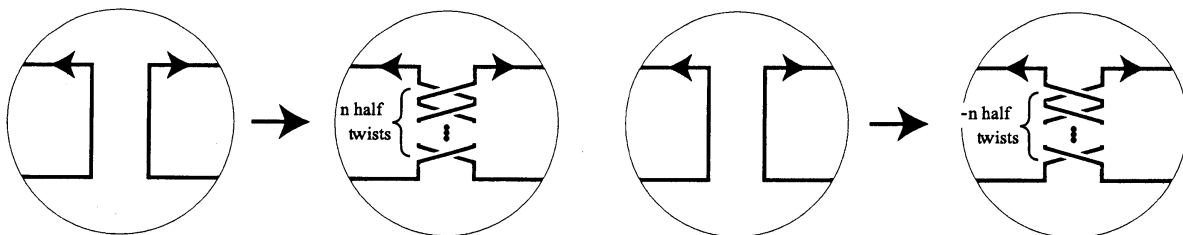
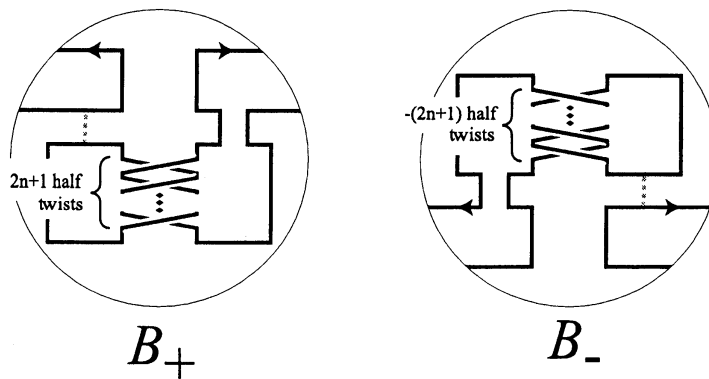


FIGURE 3. A ribbon presentation of Omae's knot.

FIGURE 4. The definition of a  $t_n$ -move for  $n > 0$  (left) and for  $n < 0$  (right).

$(B_-, T_-)$  with  $B_+ \cap B_- = \emptyset$  such that if we apply a  $t_{2n+1}$ -move for  $(B_+, T_+)$  and a  $t_{-(2n+1)}$ -move for  $(B_-, T_-)$ , then we obtain the 3-component unlink. Now we consider  $K \# T(2, 2n+1) \# T(2, -(2n+1))$  as in Figure 5. If we add

FIGURE 5. The knot  $K \# T(2, 2n+1) \# T(2, -(2n+1))$ .

two bands along dotted arcs in Figure 5, then the resulting 3-component link is trivial by the assumption. Therefore  $K \# T(2, 2n+1) \# T(2, -(2n+1))$  is ribbon.  $\square$

Now we prove Theorem 2.1.

*Proof of Theorem 2.1.* By the assumption, there exist two trivial tangles  $(B_+, T_+)$  and  $(B_-, T_-)$  with  $B_+ \cap B_- = \emptyset$  such that if we apply a  $t_{2n+1}$ -move for  $(B_+, T_+)$  and a  $t_{-(2n+1)}$ -move for  $(B_-, T_-)$ , then we obtain the 3-component unlink. If we need, by choosing another 3-balls, we may assume that two trivial tangles  $(B_+, T_+)$  and  $(B_-, T_-)$  are connected as in Figure 6. Now we consider again  $K \# T(2, 2n+1) \# T(2, -(2n+1))$  as in Figure 5 with

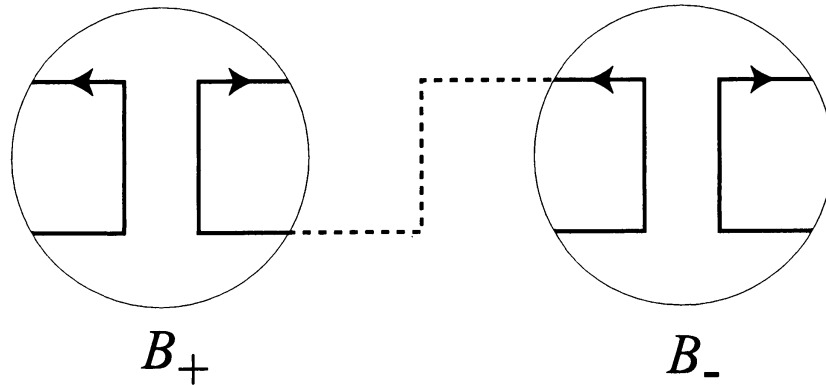


FIGURE 6. Connectivity of two trivial tangles  $(B_+, T_+)$  and  $(B_-, T_-)$ .

two bands attached along dotted arcs. Then we deform  $T(2, -(2n+1))$  as in Figure 7 with the band. We can see the knot  $T(2, 2n+1) \# T(2, -(2n+1))$

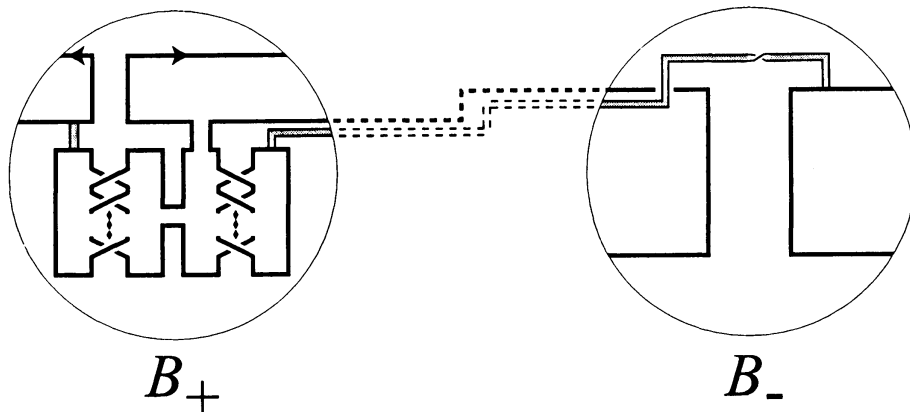
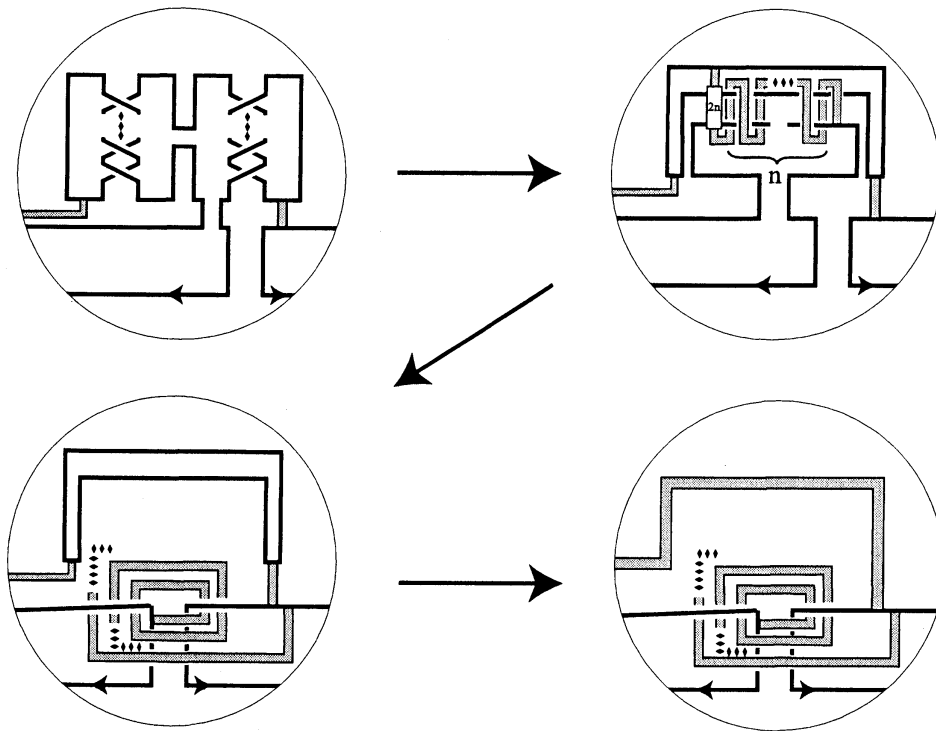


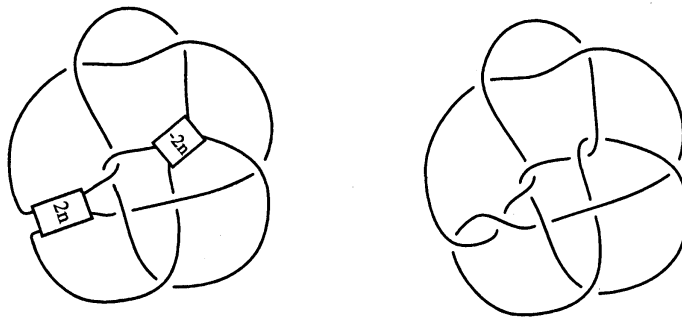
FIGURE 7. A deformation of  $T(2, -(2n+1))$ .

in  $B_+$  which is known to be ribbon. We concentrate on  $B_+$  and deform the tangle (in  $B_+$ ) as in Figure 8. Then we obtain a ribbon presentation of  $K$ .  $\square$

FIGURE 8. Deformations in  $B_+$ .

As a corollary of Theorem 2.1, we obtain the following.

**Corollary 2.3.** *The knot  $K_n$  in the left of Figure 9 is ribbon. In particular,  $K_1 = 12_{a990}$  is ribbon.*

FIGURE 9. Left: the knot  $K_n$ , Right: the knot  $12_{a990}$ .

*Proof.* We choose two 3-balls  $B_+$  and  $B_-$  as in the left of Figure 10. We apply a  $t_{2n+1}$ -move for  $(B_+, K_n \cap B_+)$  and a  $t_{-(2n+1)}$ -move for  $(B_-, K_n \cap B_-)$ . Then we obtain the 3-component link as in the right of Figure 10 which is trivial. Therefore  $K_n$  is ribbon by Theorem 2.1.  $\square$

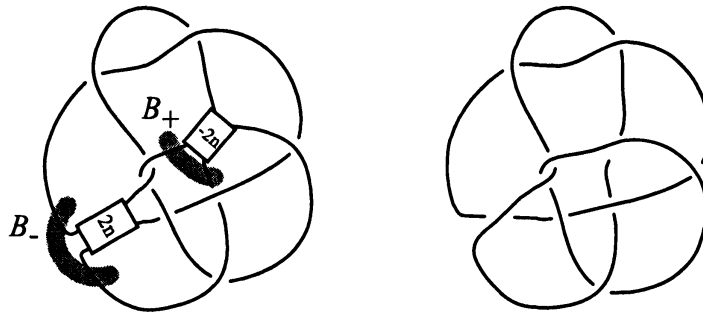


FIGURE 10. Left: 3-balls  $B_+$  and  $B_-$ , Right: the 3-component unlink.

### 3. ON THE RIBBON FUSION NUMBER

A ribbon knot  $K$  is of  $m$ -fusions if  $K$  is isotopic to

$$\bigcup_{i=0}^m S_i^1 - \text{int}\left(\bigcup_{j=1}^m b_j(\partial I \times I) \cup \bigcup_{j=1}^m b_j(I \times \partial I)\right)$$

where  $\bigcup_{i=0}^m S_i^1$  is the  $(m+1)$ -component unlink and  $b_j : I \times I \rightarrow S^3$  ( $j = 1, 2, \dots, m$ ) are disjoint embeddings such that

$$S_i^1 \cap b_j = \begin{cases} b_j(\{0\} \times I) & \text{if } i = 0, \\ b_j(\{1\} \times I) & \text{if } i = j, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is known that a ribbon knot is of  $m$ -fusions for some  $m$  [20, 25]. The *ribbon fusion number* of a ribbon knot is defined to be the minimal number of such  $m$ . For the study of the ribbon fusion number, see [3, 13, 24].

**Question 1.** *Is the ribbon fusion number of Omae's knot two ?*

**Question 2.** *Is the ribbon fusion number of the knot  $12_{a990}$  two ?*

### 4. HOMOTOPY 4-SPHERES ASSOCIATED TO UNKNOTTING NUMBER ONE RIBBON KNOTS

In the conference, Intelligence of Low-dimensional Topology, the first author talked on annulus twist, diffeomorphic 4-manifolds, and slice knots. In this section, we assume some terminologies in [1]. The first author and Jong showed the following.



**Proposition 4.1** ([1]). *Let  $K$  be an unknotting number one knot,  $(A, b, c, \varepsilon)$  the associated band presentation and  $K_n$  the knot obtained from  $K$  by applying an annulus twist  $n$  times. If  $K$  is ribbon, then there exists a homotopy 4-ball  $W_n$  with  $\partial W_n = S^3$  such that  $K_n$  bounds a smoothly embedded disk in  $W_n$ . In particular, we can associate a homotopy 4-sphere for each  $n$ .*

Let  $K$  be the knot  $8_{20}$ . Note that the unknotting number of  $8_{20}$  is one and the associated band presentation of  $K$  is depicted in Figure 11. Let  $K_n$  the knot obtained from  $K$  by applying an annulus twist  $n$  times. Then  $K_1$  is Omae's knot. Since  $8_{20}$  is ribbon, we can associate a homotopy 4-sphere  $\Sigma_n$  for each  $n$  by Proposition 4.1. Theorem 1.1 implies that  $\Sigma_1$  is standard.

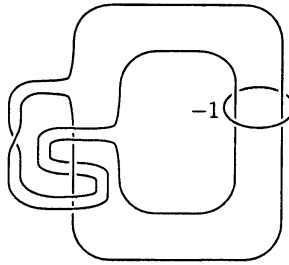


FIGURE 11. The associated band presentation for  $8_{20}$ .

**Conjecture 4.2.** *The homotopy 4-sphere  $\Sigma_n$  is standard for each  $n$ .*

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